1. Calculate the following real integrals (you might want to use the residue theorem for that nevertheless):

(a) $\displaystyle\int_0^{2\pi} \frac{\cos(\theta)\sin(2\theta)}{5+3\cos(2\theta)}\,d\theta.$

(b) $\displaystyle\int_0^{2\pi} \frac{\sin^2(\frac{5\theta}{2})}{\sin^2(\frac{\theta}{2})}\,d\theta.$

Hint: In order to avoid obtaining expressions of the form $z^{\frac{1}{2}}$ *in the complex integral, you might want to use some trigonometric identities to substitute* $\sin^2(w)$ *with a formula involving* $\cos(2w)$.

(c) For $0 < p < 1$: $\displaystyle\int_0^{2\pi} \frac{1}{1-2p\cos(\theta)+p^2}\,d\theta.$

2. Use the residue theorem to compute the Fourier transform $\hat{f}(a)$ of the function

$$f(x) = \frac{x}{1+x^4}.$$

3. In this exercise, we will show that the Fourier transform of a Gaussian function is again a Gaussian function. Let

$$f(x) = e^{-\frac{x^2}{2}}.$$

(a) Using an appropriate change of variables, show that

$$\hat{f}(a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-iax}e^{-\frac{x^2}{2}}\,dx = \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}}\lim_{L\to+\infty}\int_{-L+ia}^{L+ia} e^{-\frac{z^2}{2}}\,dz,$$

where the final integral is considered over the line $\mathrm{Im}(z) = a$ in the complex plane.

(b) Show, using the techniques that we learned about moving the curve of integration in the complex plane for integrals of holomorphic functions, that

$$\lim_{L\to+\infty}\int_{-L+ia}^{L+ia} e^{-\frac{z^2}{2}}\,dz = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\,dx.$$

Hint: In the case $a \geqslant 0$ (the case $a < 0$ is completely analogous), you might want to apply Cauchy's theorem to the rectangle which is the boundary of $\{-L \leqslant \mathrm{Re}(z) \leqslant L\} \cap \{0 \leqslant \mathrm{Im}(z) \leqslant a\}$. What happens to the integral over the edges at $\mathrm{Re}(z) = \pm L$ as $L \to +\infty$?

(c) Use polar coordinates to compute the integral

$$\Big(\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\,dx\Big)^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{x^2+y^2}{2}}\,dxdy.$$

(d) Combining the above calculations, show that

$$\hat{f}(a) = f(a).$$

(e) Using the general properties of the Fourier transform, show that, for any $\sigma \neq 0$,

$$\mathcal{F}[e^{-\frac{x^2}{2\sigma^2}}](a) = |\sigma| e^{-\frac{\sigma^2 a^2}{2}}.$$

(4.) Let $f : [0, +\infty) \to \mathbb{C}$ be a piecewise continuous function.

(a) Suppose that there exists some $\gamma_0 \in \mathbb{R}$ such that we have

$$\int_0^{+\infty} |f(t)| e^{-\gamma_0 t}\,dt < +\infty.$$

Show that, for any $\gamma > \gamma_0$, we have for the function $t \cdot f(t)$:

$$\int_0^{+\infty} |t \cdot f(t)| e^{-\gamma t}\,dt < +\infty.$$

(Hint: Show that $t \leqslant Ce^{(\gamma-\gamma_0)t}$ for some constant $C > 0$ independent of t.)

(b) Recall that a $\gamma_0 \in \mathbb{R}$ is called an *abscissa of convergence* for the Laplace transform $\mathcal{L}[f](z)$ of f if, for every $z \in \mathbb{C}$ with $\mathrm{Re}(z) > \gamma_0$, the integral defining $\mathcal{L}[f](z)$ is well-defined ("converges absolutely"), that is to say:

$$\int_0^{+\infty} |f(t)||e^{-zt}|\,dt < +\infty.$$

Using the previous part of the exercise, show that, if γ_0 is an abscissa of convergence for $\mathcal{L}[f(t)]$, then γ_0 is also an abscissa of convergence for $\mathcal{L}[t \cdot f(t)]$ (and, therefore, by repeating the same process, also for $\mathcal{L}[t^n f(t)]$ for any $n \in \mathbb{N}$).